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Coherent States on Lie Algebras: A Constructive Approach

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We generalize the notion of coherent states to arbitrary Lie algebras by making an analogy with the GNS construction in C^*-algebras. The method is illustrated with examples of semisimple and nonsemisimple finite-dimensional Lie algebras as well as loop and Kac–Moody algebras. A deformed addition on the parameter space is also introduced simplifying some expressions and some applications to conformal field theory are pointed out, e.g., differential operator and free field realizations found.

1. INTRODUCTION

For the harmonic oscillator one can define coherent states, i.e., states which are eigenstates of the creation operator $a^\dagger$ (see, e.g., Itzykson and Zuber 1985). These are given by (note: unnormalized!)

$$|z\rangle := e^{za^\dagger}|0\rangle \tag{1}$$

where $|0\rangle$ is the vacuum state (the zero-particle state of the Fock space) and z is some arbitrary complex number. These states are overcomplete

$$\langle z|z'\rangle := p(\bar{z}, z') = e^{-\bar{z}z'/2} \tag{2}$$

where $\bar{z}$ is the complex conjugate of $z \in \mathbb{C}$. We can then normalize by dividing $|z\rangle$ by $\sqrt{p(\bar{z}, z)} = \exp(-\frac{1}{4}|z|^2)$. The relevance of these states lies in their intimate connection with functional integrals. Given an operator A, we can construct its Bargmann kernel $\tilde{A}$, which is then a function of two complex variables

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$$\tilde{A}(z, z'):= \frac{\langle z|A|z'\rangle}{\langle z|z'\rangle} \tag{3}$$

and a functional integral is then defined as

$$\int \exp\Bigg[\frac{1}{2}(\bar{z}_f z'_f + \bar{z}_i z'_i) + \int_{t_i}^{t_f}\left(\frac{1}{2i}(\bar{z}\dot{z}' - \dot{\bar{z}}z') + \tilde{H}(z, z')\right) dt\Bigg] \mathfrak{D}(z, z') := U(z_f, t_f; z_i, t_i) \tag{4}$$

where $U(z_f, t_f; z_i, t_i)$ is the time-development operator and where the measure is defined as the limit $\mathfrak{D}(z, z') = \lim_{n\to\infty}(dz_n dz'_n/2\pi i)$.

For fermionic degrees of freedom, one would define coherent states in a similar way, but with the complex parameter z replaced by a Grassmann number η (Itzykson and Zuber, 1985).

Now, the harmonic oscillator is but one particularly simple example of a physical system described by a Lie algebra. In this case the algebra is $A_1 \simeq sl_2 \simeq su_2 \simeq so_3$,

$$[a^\dagger, a] = n, \qquad [n, a^\dagger] = a^\dagger, \qquad [n, a] = -a \tag{5}$$

which is the simplest nontrivial semisimple Lie algebra.

Generalizations to other semisimple Lie algebras have been made (Klauder and Skagerstam, 1985). One considers a (usually compact) Lie group G acting on some space X. Starting with a fiducial vector $|x\rangle$, $x \in X$, one defines $|g\rangle = \exp(T(g))|x\rangle$, where T is the appropriate representation. The geometric setting for this is the Borel–Weil–Bott construction (see, e.g., Nash, 1991). One first considers G as a fiber bundle over G/H with fiber H, and then constructs a holomorphic line bundle L_λ from a map $\lambda: H \to S^1$, λ a highest weight. The Peter–Weyl theorem then states that $L^2(G) \simeq \bigoplus_\lambda V_\lambda \otimes V^*_\lambda$, where V_λ denotes the set of cross sections of the line bundle L_λ, i.e., $V_\lambda = \Gamma(L_\lambda)$. The functions in V_λ are annihilated by elements of G_-, the Lie group of the algebra $\mathfrak{g}_-$ given by a root decomposition with respect to $\mathfrak{h}$, the Lie algebra of H (the Cartan algebra).

We want to propose a simple, constructive, and natural procedure which applies to nonsemisimple Lie algebras and to Kac–Moody algebras, too. Let us note that the definition of a coherent state depended on the following ingredients: (1) a root decomposition (in order to specify the creation operators), (2) a representation, and (3) a vacuum state $|0\rangle$ in the corresponding vector space (the Fock space). It is natural to attempt to construct all of this out of the structure of the algebra itself. In this way it becomes similar to the GNS construction known from operator algebras, in which one uses the structure of the algebra (C^* or just Banach) $\mathscr{A}$ to construct a natural Hilbert space $\mathcal{H}$ with a natural cyclic vector (i.e., the vacuum state) denoted by ξ, such that $\mathcal{H} = \mathcal{A}\xi$ (the algebra generates the Hilbert space) and $\mathcal{A}$ is isomorphic to a subalgebra of $\mathcal{B}(\mathcal{H})$, the algebra of bounded operators on $\mathcal{H}$. See, e.g., Murphy (1990) and Wegge-Olsen (1993).

2. THE CONSTRUCTION

We will generalize the root decomposition in the following way. Suppose we can write the Lie algebra (as a vector space) as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha \oplus \sum_{\alpha \in \Delta_-} \mathfrak{g}_\alpha \tag{6}$$

with

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}, \qquad \alpha + \beta \neq 0 \tag{7}$$

$$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{g}_0 \tag{8}$$

$$[\mathfrak{g}_0, \mathfrak{g}_0] = 0 \tag{9}$$

where α, β are elements of some vector space of dimension $\geq$dim $\mathfrak{g}_0$. We do not require dim $\mathfrak{g}_\alpha \leq 1$, nor do we require $n\alpha \in \Delta_+ \cup \Delta_- \cup \{0\} \equiv \Delta \Rightarrow n = \pm 1, 0$. Hence we will allow roots α without a corresponding mirror image $-\alpha$, or with, e.g., 2α also a root. We will also allow more than one linearly independent generator in each $\mathfrak{g}_\alpha$.

Roots which satisfy the usual requirements (each root space having dimension one, and $n\alpha$ a root only if $n = \pm 1, 0$) will be called *proper*, and will thus generate a semisimple subalgebra, whereas the remaining roots will be called *pseudo roots*. For Kac–Moody algebras the real roots are then proper, whereas the imaginary ones are pseudo roots (but each space $\mathfrak{g}_\alpha$ is one dimensional). In this case the proper roots span the corresponding finite-dimensional Lie algebra.

As for semisimple Lie algebras, we will draw the roots as vectors in some (for finite-dimensional algebras) finite-dimensional space. If there is more than one independent generator in a given $\mathfrak{g}_\alpha$, then the corresponding arrow is drawn differently: if dim $\mathfrak{g}_\alpha = 2$, then we will draw the arrow as $\upuparrows$, whereas for dim $\mathfrak{g}_\alpha \geq 3$, we will include the dimensionality as a subscript, $\upuparrows_d$, with $d =$ dim $\mathfrak{g}_\alpha$.

Let us consider some examples. The trivial Lie algebra $\mathbb{F}$ where $\mathbb{F}$ is some field (e.g., $\mathbb{F} = \mathbb{R}, \mathbb{C}$) is then drawn as a simple arrow $\uparrow$, whereas $\mathbb{F}^2$ is drawn as $\upuparrows$. These are of course Abelian. For an example of a non-Abelian algebra, consider the Heisenberg algebra in a one-dimensional space h_1; this is drawn as

$$\begin{array}{c}\uparrow\\ \Uparrow\end{array}$$

where the generators of $\Uparrow$ are denoted by q, p and where the generator of the uppermost arrow is $i\hbar 1$. This corresponds to a decomposition

$$h_1 = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \tag{10}$$

where dim $\mathfrak{g}_1 = 2$ and dim $\mathfrak{g}_2 = 1$, $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$. Here $\mathfrak{g}_0 \equiv 0$ and there are *no* negative roots; all roots are pseudo roots, denoted by 1, 2. One should note that for nonsemisimple Lie algebras the root decomposition will in general be nonunique. An alternative decomposition for h_1 would have been $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with each component being one dimensional; $\mathfrak{g}_{-1} = \mathbb{F}_p$, $\mathfrak{g}_0 = \mathbb{F}(i\hbar 1)$, $\mathfrak{g}_1 = \mathbb{F}q$. This latter choice, however, would obscure the very strong difference between the nilpotent algebra h_1 and the semisimple one A_1. We have chosen the decomposition which most clearly brings out this difference between the two algebras.

When $\mathfrak{g}_0 \neq 0$ but all roots are still pseudo, we will denote the elements of $\mathfrak{g}_0$ by a circle. Consider, for instance, the unique two-dimensional non-Abelian algebra $[e, f] = e$. This has a decomposition $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_0 = \mathbb{F}f$ and $\mathfrak{g}_1 = \mathbb{F}e$. The root diagram is

$$\begin{array}{c}\uparrow\\ \circ\end{array}$$

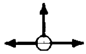
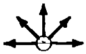
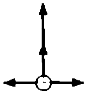
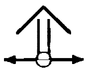
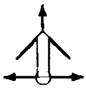
We conjecture that all finite-dimensional Lie algebras can be treated in this manner.

Consider extensions of a semisimple Lie algebra, say $A_1 = sl_2 = su_2 = so_3$ for simplicity. Some of the ways of extending it by pseudo roots are shown in Table I. The Jacobi identity fixes most of the algebraic relations uniquely, and the corresponding Lie algebras are listed in the table, too.

The second example in the table, the one where the new generators are e_s, $e_{s\pm r}$, will be called the *fan algebra*, because of the shape of the root diagram, and will be the standard example together with the Heisenberg algebra of a nonsemisimple Lie algebra. We will denote the fan algebra by $\mathfrak{f}_3(A_1)$ or just $\mathfrak{f}_3$, the subscript 3 referring to the three extra roots we have added to A_1, namely e_s, $e_{s\pm r}$. Similarly one can define $\mathfrak{f}_{2n+1}(A_1)$ for $n \geq 1$.

Now given a Lie algebra $\mathfrak{g}$, in order to define coherent states, we must first of all find a natural vector space for it to act upon. The obvious choice is the underlying vector space of the algebra, i.e., the algebra itself. The corresponding representation is the adjoint one. Furthermore, the roots (proper as well as pseudo) lying in Δ_+ are the natural candidates for creation operators. Note, however, that for pseudo roots it is purely a matter of convention whether one includes a root in Δ_+ or in Δ_-. The two different choices are each other's duals.

Table I. The First Nonsemisimple Lie Algebras Which Can Be Built from A_1 by Adding Pseudo Roots.[a]

diagram	algebra
	$\mathfrak{g} \simeq A_1 \oplus \mathbb{F}e_s$
	$[e_s, e_{\pm r}] = N_{s,\pm r} e_{s\pm r}$ $e_s \in Z(\mathfrak{g})$
	$\mathfrak{g} \simeq A_1 \oplus \mathbb{F}e_s \oplus \mathbb{F}e_{2s}$
	$\mathfrak{g} \simeq A_1 \oplus (\mathbb{F}e_s)^2$ $[e_s^{(1)}, e_s^{(2)}] = 0$
	$\mathfrak{g} \simeq A_1 \oplus h_1$

[a] The $\oplus$ denotes direct sum as Lie algebras and not just (as in the text) as vector spaces.

The basic ingredient is then the element

$$x_\alpha(\zeta) = e^{\zeta \mathrm{ad} e_\alpha}, \qquad \alpha \in \Delta_+ \tag{11}$$

It turns out that this quantity is important in its own right, as it generates what is known as the Chevalley group (Carter,1989). In order to define a vector $|\zeta\rangle$ we must specify a "vacuum state" $|0\rangle := v_0$. This state, in analogy with the cyclic vector of the GNS construction, must satisfy

$$\mathrm{ad}\; e_\alpha v_0 = 0 \qquad \text{for} \quad \alpha \in \Delta_- \tag{12}$$

$$\mathrm{span}\{x_\alpha(\zeta) v_0\}_{\alpha \in \Delta_+} = \mathfrak{g} \qquad \text{(as a vector space)} \tag{13}$$

$$\mathrm{ad}\; h_i v_0 = \lambda_i v_0, \qquad h_i \in \mathfrak{g}_0 \tag{14}$$

In words: the vacuum is annihilated by the elements corresponding to negative roots (the annihilation operators), is an eigenvector of elements of $\mathfrak{g}_0$ (the generalized Cartan algebra, the "number operators"), and generates the entire vector space when acted upon by elements of $\mathfrak{g}_\alpha$, $\alpha \in \Delta_+$. This is the Lie algebra analogue of the GNS construction for operator algebras.

We then define the coherent states as

$$|\zeta\rangle := \exp\,(\sum_{\alpha \in \Delta_+} \zeta_\alpha \;\mathrm{ad}\; e_\alpha) v_0 \tag{15}$$

where $\zeta = (\zeta_\alpha) \in \mathbb{F}^{|\Delta_+|}$. For the "dual" element, the bra $\langle\zeta|$, there are two, in general, inequivalent possibilities. One uses the generalized Chevalley involution[2]

$$\{e_\alpha\} \to \{-f_\beta\} \qquad h_i \to -h_i \tag{16}$$

with $\alpha \in \Delta_+$, $\beta \in \Delta_-$, $i = 1, \ldots, \dim \mathfrak{g}_0$. Since for $\mathfrak{g}$ nonsemisimple, $|\Delta_+| > |\Delta_-|$ this "involution" is not bijective. The other possibility is to let $\langle\zeta|$ be simply the complex conjugate transpose of $|\zeta\rangle$, i.e.,

$$\langle\zeta'| := v_0^t \exp(-\sum_{\alpha\in\Delta_+} \bar{\zeta}'_\alpha \operatorname{ad} e_\alpha)^t \tag{17}$$

where the superscript t denotes transpose. This is the definition we will choose. For semisimple Lie algebras the two definitions coincide.

These coherent states are overcomplete and we define

$$p(\bar{\zeta}, \zeta') := \langle\zeta|\zeta'\rangle \tag{18}$$

Then p is some polynomial when the algebra is semisimple and a holomorphic function otherwise (for semisimple Lie algebras the adjoint representation is nilpotent, so the exponentials are finite-order polynomials). One should also note that the coherent states are not normalized. This can simply be done by dividing by $\sqrt{p(\bar{\zeta}, \zeta)}$.

A particularly important subject to study is central extensions. Suppose we have a Lie algebra $\mathfrak{g}$, and then form a central extension $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{F}c$; we then would like to know how coherent states for $\mathfrak{g}$ are related to those of $\tilde{\mathfrak{g}}$. Write the algebraic relations of the centrally extended algebra as

$$\begin{aligned}
[e_\alpha, e_\beta] &= N_{\alpha,\beta} e_{\alpha+\beta} + c_{\alpha\beta} \\
[e_\alpha, f_\alpha] &= \alpha^i h_i + c_{\alpha,-\alpha} \\
[e_\alpha, h_i] &= -\alpha_i e_\alpha + c_{\alpha i} \\
[h_i, h_j] &= c_{ij}
\end{aligned}$$

etc.; then the adjoint representation becomes

$$\operatorname{ad} e_\alpha = \begin{pmatrix} \underline{\operatorname{ad}} e_\alpha|_0 & 0 \\ c_\alpha & 0 \end{pmatrix}, \qquad \operatorname{ad} f_\alpha = \begin{pmatrix} \underline{\operatorname{ad}} f_\alpha|_0 & 0 \\ c_{-\alpha} & 0 \end{pmatrix}, \qquad \operatorname{ad} h_i = \begin{pmatrix} \underline{\operatorname{ad}} h_i|_0 & 0 \\ c_i & 0 \end{pmatrix} \tag{19}$$

[2] We use the following convention: e_α corresponds to a creation operator, i.e., $\alpha \in \Delta_+$, whereas f_α are the annihilation operators, i.e., $\alpha \in \Delta_-$. For proper roots we have a Chevalley involution ω: $e_\alpha \leftrightarrow -f_{-\alpha}$, $h_i \mapsto -h_i$. Furthermore, we will always choose $|\Delta_+| \geq |\Delta_-|$, i.e., pseudo roots without a mirror image will be considered positive.

where $(\bar{c}_\alpha)_\beta = c_{\alpha\beta}$ and where ad $e_\alpha|_0$ denotes the matrix representing ad e_α in $\mathfrak{g}$.

Writing the new vacuum vector as $\bar{v}_0 = (v_0, 0)$, we get

$$|\zeta\rangle = |\zeta_0 + \sum_{\alpha,\beta} \zeta^\alpha c_{\alpha,\beta} v_0^\beta |c\rangle := |\zeta\rangle_0 + c(\zeta, v_0)|c\rangle \tag{20}$$

where we have defined $|c\rangle$ as the basis vector of $\tilde{\mathfrak{g}}$ (as a vector space) which is in the direction of the central element c, and where $|\zeta\rangle_0$ denotes the coherent states of $\mathfrak{g}$. Since c is a central element, it follows that

$$p(\bar{\zeta}, \zeta') := \langle\zeta|\zeta'\rangle = p_0(\bar{\zeta}, \zeta') + c(\zeta, v_0)^* c(\zeta', v_0) \tag{21}$$

in the obvious notation, where p_0 is the normalization polynomial of $\mathfrak{g}$. We will now consider some examples.

2.1. Semisimple Lie Algebras

We will explicitly construct the coherent states for all four seimisimple Lie algebras of rank at most two, i.e., A_1, A_2, B_2, G_2, and also make some general statements.

The Lie algebra $A_1 \simeq sl_2 \simeq su_2 \simeq so_3$ is very quickly treated. We have $(\mathrm{ad}\ e)^2 = (\mathrm{ad}\ f)^2 = 0$, so ($\omega$ denoting the Chevalley involution)

$$x(\zeta) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2}\zeta^2 & 1 & \zeta \\ -\zeta & 0 & 1 \end{pmatrix}, \qquad \omega^* x(\bar{\zeta}) := e^{-\bar{\zeta}\mathrm{ad} f} = \begin{pmatrix} 1 & -\frac{1}{2}\bar{\zeta}^2 & \bar{\zeta} \\ 0 & 1 & 0 \\ 0 & -\bar{\zeta} & 1 \end{pmatrix} \tag{22}$$

The eigenvector of ad h annihilated by ad f is $v = (1, 0, 0)$ (with the weight $\lambda = 2$), which leads to the following set of coherent states:

$$|\zeta\rangle = v + \frac{1}{2}\zeta^2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \zeta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{23}$$

which we will also write as

$$|\zeta\rangle = |1\rangle + \frac{1}{2}\zeta^2|2\rangle - \zeta|3\rangle \tag{24}$$

with $|i\rangle$ being the ith canonical basis vector for $\mathbb{F}^3$ (i.e., $\mathfrak{g}$ considered as a vector space). The dual state $\langle\zeta|$ is obtained from this by making the substitutions $\zeta \to -\bar{\zeta}$ and $|i\rangle \to \langle i|$, i.e.,

$$\langle\zeta| = \langle 1| + \frac{1}{2}\bar{\zeta}^2\langle 2| + \bar{\zeta}\langle 3| \tag{25}$$

The quadratic (in each variable, i.e., quartic in all) polynomial for the normalization becomes

$$\langle\zeta|\zeta'\rangle = p(\bar{\zeta}, \zeta') = 1 - \bar{\zeta}\zeta' + \frac{1}{4}\bar{\zeta}^2\zeta'^2 \tag{26}$$

as one quickly sees.

This is not the same as the standard coherent states for the harmonic oscillator (1) because we are using the adjoint representations, which is nilpotent, i.e., $\exists p$: $(a\dagger)^p = 0$. The standard coherent states corresponds to $p = \infty$, which makes the algebra into a C^*-algebra (the bilateral shift algebra). The next algebras are the three rank-two simple Lie algebras $A_2 \simeq su_3$, $B_2 \simeq so_4$, G_2. In these simple cases we can actually also compute the exponential of the adjoint representation quite easily.

For $A_2 \simeq su_3 \simeq sl_3$ we have the following simple roots: $\pm r$, $\pm s$, $\pm(r + s)$, and the two Cartan elements h_r, h_s. In this case

$$(\text{ad } e_r)^3 = (\text{ad } e_s)^4 = (\text{ad } e_{r+s})^3 = 0 \tag{27}$$

The vacuum vector is $v = (0, 0, 1, 0, 0, 0, 0, 0) = |3\rangle$ with weight $\lambda = (1, 1)$, and we get

$$|\zeta\rangle := (e^{\zeta_r \text{ad} e_r + \zeta_s \text{ad} e_s + \zeta_r{}^{+s} \text{ad} e_r{}^{+s}}) \cdot v \tag{28}$$

$$\begin{aligned}
&= -N_{r,s}\,\zeta_s|1\rangle + N_{r,s}\zeta_r|2\rangle + |3\rangle \\
&\quad + \left(\frac{1}{6}\,\zeta_r(2N_{r,s}\zeta_s(2\zeta_r - \zeta_s) + 3\zeta_{r+s}(N_{r,s}N_{-r,r+s} - 2))\right)|4\rangle \\
&\quad + \left(\frac{1}{6}\,\zeta_s(N_{r,s}\zeta_s(2\zeta_s - \zeta_r) - 3\zeta_{r+s}(1 + N_{r,s}N_{-s,r+s}))\right)|5\rangle \\
&\quad + \left[\frac{1}{24}\,N_{r,s}\zeta_r\zeta_s^2(\zeta_r(4N_{s,-r-s} - N_{r,-r-s}) + 2\zeta_s(N_{r,-r-s} - N_{s,-r-s}))\right. \\
&\quad + \frac{1}{6}\,\zeta_s\zeta_{r+s}(\zeta_r(N_{r,s} - N_{r,-r-s}\,(1 - N_{r,s}\,N_{-s,r+s}) \\
&\quad \left. - N_{s,-r-s}(2 + N_{r,s}N_{-s,r+s})) - \zeta_{r+s}^3)\right]|6\rangle \\
&\quad + \left[N_{r,s}\zeta_s\left(\zeta_r - \frac{1}{2}\,\zeta_s\right) - \zeta_{r+s}\right]|7\rangle - \left[N_{r,s}\zeta_s\left(\frac{1}{2}\,\zeta_r - \zeta_s\right) + \zeta_{r+s}\right]|8\rangle
\end{aligned} \tag{29}$$

The normalization of the coherent states become

$$
\begin{aligned}
p(\bar{\zeta}, \zeta') := {}& 1 - N_{-r,r+s}N_{r,s}\bar{\zeta}_r\zeta'_r + N_{-s,r+s}N_{r,s}\bar{\zeta}_s\zeta'_r \\
&+ \left(\bar{\zeta}_{r+s} - \frac{1}{2}N_{-s,r+s}\bar{\zeta}_r\bar{\zeta}_s\right)\left(N_{r,s}\zeta'_r\zeta'_s - \frac{1}{2}N_{r,s}\zeta'^2_s - \zeta'_{r+s}\right) \\
&- \left(\bar{\zeta}_{r+s} - \frac{1}{2}N_{-r,r+s}\bar{\zeta}_r\bar{\zeta}_s\right)\left(\frac{1}{2}N_{r,s}\zeta'_r\zeta'_s - N_{r,s}\zeta'^2_s + \zeta'_{r+s}\right) \\
&- \frac{1}{36}\bar{\zeta}_r\zeta'_r((N_{-r,r+s} - 2N_{-s,r+s})\bar{\zeta}_r\bar{\zeta}_s + 3\bar{\zeta}_{r+s}(1 + N_{-r,r+s}N_{s,-r-s})) \\
&\times (2N_{r,s}\zeta'_s(2\zeta'_r - \zeta'_s) - 3\zeta'_{r+s}(N_{r,s}N_{-r,r+s} + 2)) \\
&+ \frac{1}{36}\bar{\zeta}_s\zeta'_s(\bar{\zeta}_r\bar{\zeta}_s(N_{-s,r+s} - 2N_{-r,r+s}) + 3\bar{\zeta}_{r+s}(1 + N_{-s,r+s}N_{r,-r-s})) \\
&\times (N_{r,s}\zeta'_s(\zeta'_r - 2\zeta'_s) + 3\zeta'_{r+s}(1 - N_{r,s}N_{-s,r+s})) \\
&+ \Bigg[\frac{1}{8}\bar{\zeta}^2_r\bar{\zeta}^2_s N_{-r,-s}(N_{-s,r+s} - N_{-r,r+s} \\
&+ \frac{1}{6}\bar{\zeta}_r\bar{\zeta}_s\bar{\zeta}_{r+s}(N_{-r,r+s}(1 + N_{-r,-s}N_{s,-r-s}) \\
&+ N_{-s,r+s}(1 + N_{-r,-s}N_{r,-r-s})) - \bar{\zeta}^3_{r+s}\Bigg) \\
&\times \Bigg(\frac{1}{24}N_{r,s}\zeta'_r\zeta'^2_s(\zeta'_r(4N_{-s,r+s} - N_{r,-r-s}) + 2\zeta'_s(N_{r,-r-s} - N_{s,-r-s})) \\
&+ \frac{1}{6}\zeta'_s\zeta'_{r+s}(\zeta'_r(N_{r,s} - N_{r,-r-s}(1 - N_{r,s}N_{-s,r+s}) \\
&- N_{s,-r-s}(2 - N_{r,s}N_{s,-r-s})) + N_{r,s}\zeta'_s\zeta'^2_{r+s})\Bigg)
\end{aligned} \tag{30}
$$

which is a polynomial of sixth degree with the ζ' and $\bar{\zeta}$ variables appearing to at most the third power.

For $B_2 \simeq so(4)$ we have the roots $\pm r$, $\pm s$, $\pm(r + s)$, $\pm(2r + s)$, the Cartan elements once more denoted by h_r, h_s. Thus

$$(\text{ad } e_r)^4 = (\text{ad } e_s)^3 = (\text{ad } e_{r+s})^4 = (\text{ad } e_{2r+s})^3 = 0 \tag{31}$$

Then $(\Sigma_{\alpha>0}\,\zeta^\alpha \text{ ad } e_\alpha)^8 = 0$ and the exponential becomes easy to calculate. The lowest weight is $\lambda = (-6, 5)$ and the corresponding "vacuum" vector is $v = (0, 0, 0, 1, 0, 0, 0, 0, 0) = |4\rangle$; one easily checks that this is annihilated by the ad f terms. The coherent states thus become

$$
\begin{aligned}
|\zeta\rangle = &-\frac{1}{2} N_{r,r+s}(N_{r,s}\zeta_r\zeta_s + 2\zeta_{r+s})|1\rangle \\
&+ \frac{1}{2} N_{r,s}N_{r,r+s}\zeta_r^2|2\rangle + N_{r,r+s}\zeta_r|3\rangle + |4\rangle \\
&+ \frac{1}{6}\zeta_r[N_{r,s}N_{r,r+s}\zeta_r^2\zeta_s + N_{r,r+s}(N_{r,s}N_{r+s,-r} - 1)\zeta_r\zeta_{r+s} \\
&+ (15 + 3N_{r,r+s}N_{2r+s,-r})\zeta_{2r+s}]|5\rangle \\
&- \left[\frac{1}{8} N_{r,s}N_{r,r+s}\zeta_r^2\zeta_s^2 - \frac{1}{6} N_{r,r+s}(3 - N_{r,s}N_{r+s,-s})\zeta_r\zeta_s\zeta_{r+s}\right. \\
&\left. + \frac{1}{2} N_{r,r+s}N_{r+s,-s}\zeta_{r+s}^2 - 3\zeta_s\zeta_{2r+s}\right]|6\rangle \\
&+ \left[\frac{1}{120} N_{r,r+s}N_{r,s}(4N_{s,-r-s} - 3N_{r,-r-s})\zeta_r^3\zeta_s^2\right. \\
&+ \frac{1}{24} N_{r,r+s}(N_{r,s} + N_{r,-r-s}(3 - N_{r,s}N_{r+s,-s}) \\
&+ N_{s,-r-s}(N_{r,s}N_{r+s,-r} - 1))\zeta_r^2\zeta_s\zeta_{r+s} \\
&+ \left(N_{r,-r-s} - \frac{5}{6} N_{s,-r-s} + \frac{1}{6}(N_{r,r+s}N_{s,-r-s}N_{2r+s,-r}\right. \\
&\left. - N_{r,r+s}N_{r,s}N_{2r+s,-r-s})\right)\zeta_s\zeta_{2r} \\
&+ \frac{1}{2}(1 - N_{r,r+s}N_{2r+s,-r-s})\zeta_{r+s}\zeta_{2r+s} \\
&\left. + \frac{1}{6} N_{r,r+s}(2 - N_{r,-r-s}N_{r+s,-s})\zeta_{r+s}^3\right]\zeta_r|7\rangle \\
&+ \left[\frac{1}{720} N_{r,-2r-s}N_{r,r+s}N_{r,s}(4N_{s,-r-s} - 3N_{r,-r-s})\zeta_r^4\zeta_s^2\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{120} N_{r,r+s}(N_{r,-2r-s}(3N_{r,-r-s} + N_{r,s} - N_{r,-r-s}N_{r,s}N_{r+s,-s} \\
&- N_{s,-r-s} + N_{r,s}N_{r+s,-r}N_{s,-r-s}) + 4N_{r,s}N_{r+s,-2r-s})\zeta_r^3\zeta_s\,\zeta_{r+s} \\
&+ \frac{1}{6}(N_{r,-2r-s}(1 - N_{r,r+s}N_{2r+s,-r-s}) \\
&+ N_{r+s,-2r-s}(N_{r,r+s}N_{2r+s,-r} - 5) + N_{r,r+s})\zeta_r\zeta_{r+s}\zeta_{2r+s} \\
&- 2\zeta_{2r+s}^4 + \frac{1}{24}(N_{r,r+s}(2N_{r,-2r-s} - N_{r,-r-s}N_{r+s,-s}N_{r,-2r-s} \\
&- N_{r+s,-2r-s} + N_{r,s}\,N_{r,-2r-s}N_{r+s,-2r-s})\zeta_{r+s}^3 \\
&+ (6N_{r,-r-s}N_{r,-2r-s} + 5N_{r,r+s}N_{r,s} - 5N_{r,-2r-s}N_{s,-r-s} \\
&+ N_{r,-2r-s}N_{r,r+s}(N_{s,-r-s}N_{2r+s,-r} - N_{r,s}N_{2r+s,-r-s})\zeta_s\zeta_{2r+s})]|8\rangle \\
&+ \left[\frac{2}{3}N_{r,s}N_{r,r+s}\zeta_r^2\zeta_s - \frac{1}{2}N_{r,r+s}\zeta_r\zeta_{r+s} - 5\zeta_{2r+s}\right]|9\rangle \\
&+ \left[\frac{3}{2}N_{r,r+s}\zeta_r\zeta_{r+s} - \frac{1}{2}N_{r,r+s}N_{r,s}\zeta_r^2\zeta_s + 6\zeta_{2r+s}\right]|10\rangle
\end{aligned}
\tag{32}
$$

From this we can get the coherent states for $so(2, 2)$ and $so(3, 1)$ by multiplying certain of the e_α, f_α by a factor i [using $so(3, 1) \simeq su_2 \otimes \mathbb{C}$, $so(2, 2) \simeq su(1, 1) \oplus su(1, 1)$ with $su(1, 1)$ obtained from su_2 by multiplying one of the e, f-generators by i].

The "dual" state $\langle\zeta|$ is, as always, found by making the substitutions $\zeta_i \to -\bar{\zeta}_i$ and replacing the kets $|i\rangle$ with the corresponding bras. We will not, however, write down the explicit formula for the normalization polynomial $p(\zeta, \zeta') = \langle\zeta|\zeta'\rangle$, as this is far too big an expression. One should note, though, that finding it is a rather easy and straightforward task (at least with a computer).

Finally, the exceptional Lie algebra G_2 has the roots $\pm r$, $\pm s$, $\pm(r \pm s)$, $\pm(r + 2s)$, $\pm(2r + s)$, whence

$$
(\text{ad } e_r)^4 = (\text{ad } e_s)^4 = (\text{ad } e_{r+s})^4 = (\text{ad } e_{r+2s})^4 = 0, \tag{33}
$$
$$
(\text{ad } e_{2r+s})^2 = (\text{ad } e_{r-s})^2 = 0
$$

The "vacuum" vector is $v = (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$, which has the weight $\lambda = (-1,2)$. An explicit calculation shows ad $f_a \cdot v = 0$, as it should be. We get

$$|\zeta\rangle = \sum_{n=1}^{14} a_n |n\rangle \tag{34}$$

where (only writing the simplest coefficients)

$$a_1 = -N_{r,r+s}\left(\frac{1}{2} N_{r,s}\zeta_r\zeta_s + \zeta_{r+s} + \frac{1}{6} N_{r,s}N_{r-s,r+2s}\zeta_s^2\zeta_{r+2s}\right)$$

$$a_2 = -\frac{1}{24} N_{r,s}N_{r,r+s}N_{r-s,r+2s}N_{r-s,r}\zeta_s^2\zeta_{r+2s}^2$$
$$+ \frac{1}{6} N_{r,s}N_{r,r+s}(N_{r-s,r+2s} - N_{r-s,r})\zeta_r\zeta_s\zeta_{r+2s}$$
$$+ \frac{1}{2} N_{r,s}N_{r,r+s}\zeta_r^2 + \frac{1}{2} (N_{r-s,r}N_{r+s,r} + N_{r+s,r+2s}N_{r-s,r+2s})\zeta_{r+s}\zeta_{r+2s}$$

$$a_3 = N_{r,r+s}\zeta_r + \frac{1}{2} N_{r,r+s}N_{r-s,r+2s}\zeta_s\zeta_{r+2s}$$

$$a_4 = 1$$

$$a_5 = N_{r-s,r+2s}\zeta_{r+2s}$$

$$a_6 = -\frac{1}{24} N_{r,r+s}N_{s,r-s}(4N_{r,s}\zeta_r\zeta_s^2 - 12\zeta_s\zeta_{r+s} + N_{r,s}N_{r-s,r+2s}\zeta_s^3\zeta_{r+2s})$$

$$\vdots$$

$$a_{13} = \zeta_{r-s} - \frac{1}{120} N_{r,s}N_{r,r+s}N_{r-s,r+2s}(N_{s,r-s} - 3N_{r,r-s})\zeta_s^3\zeta_{r+2s}^2$$
$$- \frac{5}{2} N_{r,r+s}\zeta_r\zeta_{r+s} + \frac{1}{2} N_{r-s,r+2s}\zeta_{r+2s}\zeta_{2r+s}$$
$$+ \frac{1}{24} N_{r,s}N_{r,r+s}(N_{r-s,r+2s} + 3N_{r,r-s} - N_{s,r-s})\zeta_r\zeta_s^2\zeta_{r+2s}$$
$$+ \frac{1}{6} N_{r,s}N_{r,r+s}\zeta_r^2\zeta_s + \frac{1}{6} (3N_{r-s,r+2s}(N_{r+s,r+2s} - N_{r,r+s})$$
$$- N_{r,r+s}(N_{s,r-s} - 3N_{r,r-s}))\zeta_s\zeta_{r+s}\zeta_{r+2s}$$

$$a_{14} = -2\zeta_{r-s} - \frac{1}{120} N_{r,s}N_{r,r+s}N_{r-s,r+2s}(N_{s,-r-s} + 2N_{r,r-s})\zeta_s^3\zeta_{r+2s}^2$$
$$+ 3N_{r,r+s}\zeta_r\zeta_{r+s} - \frac{7}{2} N_{r-s,r+2s}\zeta_{r+2s}\zeta_{2r+s} - \frac{1}{6} N_{r,s}N_{r,r+s}\zeta_r^2\zeta_s$$

$$- \frac{1}{24} N_{r,s} N_{r,r+s} (N_{-s,r+2s} + 2N_{r,r-s} + N_{s,r-s}) \zeta_r \zeta_s^2 \zeta_{r+2s}$$

$$+ \frac{1}{6} (N_{r-s,r+2s} (5 N_{r,r+s} - 2 N_{r+s,r+2s})$$

$$- N_{r,r+s} (N_{s,r-s} + 2 N_{r,r-s})) \zeta_s \zeta_{r+s} \zeta_{r+2s}$$

In this case the normalization polynomial becomes of fifth order in each variable, but will not be written out explicitly (the *Mathematica* output is 37 pages long!).

Before we close this section we make some general comments. The normalization polynomials can be expressed in terms of the structure constants $N_{r,s}$, the Cartan matrix A_{rs}, and the coefficients in the Baker–Campbell–Hausdorff series, which we denote by b_i. There is some subtlety involved in this, as even though $\langle \zeta|$ can be obtained from $|\zeta\rangle$ by the simple procedure $\zeta_i \to -\zeta_i$ and exchanging bras for kets, it does not follows that $\langle \zeta | \zeta' \rangle$ is the naive inner product of these two. This is so because $\omega^* x(\zeta) x(\zeta')$ can get extra contributions to its Cartan algebra-valued terms (i.e., terms proportional to ad h_i). These extra terms arise from the Baker–Campbell–Hausdorff (BCH) formula. Obviously the first contribution is from $exp(- \sum_\alpha \frac{1}{2} \zeta_\alpha \zeta'_\alpha \alpha^i \text{ ad } h_i)$, which is precisely the first term in the BCH formula, $b_1 = \frac{1}{2}$. There will also be a contribution from the next term, b_2 [[ad f_α, ad e_β], ad e_γ] + b_2 [ad f_α, [ad f_β, ad e_γ]], whenever $\gamma = \alpha - \beta$ in the first term or $\gamma = \alpha + \beta$ in the second part. The explicit form of the contribution will be $b_2(N_{-\alpha,\beta} \lambda_i(\alpha^i - \beta^i) + N_{-\beta,-\alpha} \lambda_i(\alpha^i + \beta^i))$ with $b_2 = \frac{1}{12}$. The general pattern should now be clear.

2.2. Nonsemisimple Lie Algebras

We will consider only a few examples. First the Heisenberg algebra h_1. In our notation the basis is

$$[e_1^{(i)}, e_1^{(j)}] = \varepsilon^{ij} e_2, \qquad i, j = 1, 2 \tag{35}$$

where, in standard notation, $e_1^{(1)} = \hat{q}$, $e_1^{(2)} = \hat{p}$, $e_2 = -i\hbar \hat{1}$. Thus $\Delta_+ = \{1, 2\}$ with dim $\mathfrak{g}_1 = 2$, dim $\mathfrak{g}_2 = 1$. The adjoint representation reads

$$\text{ad } e_1^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{ad } e_1^{(2)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{ad } e_2 = 0 \tag{36}$$

It is a general feature that central elements do not appear in this formalism, as they are represented by the zero matrix. The vacuum vector is $v = (0, 0, 1)$ and we have

$$|\zeta_1, \zeta_2\rangle = \exp\begin{pmatrix} 0 & 0 & \zeta_2 \\ 0 & 0 & -\zeta_1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{37}$$

$$= \begin{pmatrix} \zeta_2 \\ -\zeta_1 \\ 1 \end{pmatrix} \tag{38}$$

We get

$$p := \langle\zeta_1, \zeta_2|\zeta_1', \zeta_2'\rangle = 1 + \bar{\zeta}_1\zeta_1' + \bar{\zeta}_2\zeta_2' := 1 + \bar{\zeta} \cdot \zeta' \tag{39}$$

where we have written $\zeta = (\zeta_1, \zeta_2) \in \mathbb{C}^2$ in the last equality. Hence the norm of a coherent state is $p(\bar{\zeta}, \zeta) = 1 + \|\zeta\|^2$. Since there are no poles in this expression we can normalize the states

$$|\zeta) := \frac{|\zeta\rangle}{1 + \|\zeta\|^2} \tag{40}$$

The set of coherent states span the Hilbert space $\mathbb{R}^3 \otimes \mathbb{C}(\zeta, \bar{\zeta})$, where $\zeta \in \mathbb{C}^2$.[3] The next example is the unique non-Abelian Lie algebra of dimension two,

$$[e, h] = e \tag{41}$$

The adjoint representation reads

$$\text{ad } e = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \qquad \text{ad } h = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \tag{42}$$

from which we get

$$x(\zeta) = e^{\zeta \text{ad} e} = 1 + \zeta \text{ ad } e \tag{43}$$

[3] Standard algebraic notation: $\mathbb{F}[x]$ denotes the set of polynomials in one variable x and coefficients from the field $\mathbb{F}$, $\mathbb{F}(x)$ is the corresponding field of fractions, $\mathbb{F}(x) = \{p(x)/q(x) | p(x), q(x) \in \mathbb{F}(x), q(x) \neq 0\}$. Furthermore, $\mathbb{F}[[x]]$ denotes the set of formal power series and $\mathbb{F}((x))$ that of formal Laurent series, $\mathbb{F}((x)) = \mathbb{F}[[x, x^{-1}]]$.

With $v = (1, 0)$ we then get the coherent state

$$|\zeta\rangle = \begin{pmatrix} 1 \\ -\zeta \end{pmatrix} \tag{44}$$

from which we get

$$p(\bar{\zeta}, \zeta') = 1 + \bar{\zeta}\zeta' \tag{45}$$

This is exactly the same as for the Heisenberg algebra except that ζ is now one dimensional, $\zeta \in \mathbb{C}$, and not two dimensional.

The final example we consider is the "fan algebra" $\mathfrak{f}_3$. The algebraic relations are

$$[e_s, e_t] = \begin{cases} 0, & t = s, s \pm r \\ N_{s\pm r} e_{s\pm r}, & t = \pm r \end{cases} \tag{46}$$

$$[h, e_t] = \begin{cases} 0, & t = s, s \pm r \\ \pm 2e_{\pm r}, & t = \pm r \end{cases} \tag{47}$$

whence (the ordering being chosen to be $r, -r, s, s + r, s - r, 0$)

$$\text{ad } e_r = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -N_{s,r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{ad } e_s = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ N_{s,r} & 0 & 0 & 0 & 0 & 0 \\ 0 & N_{s,-r} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{48}$$

$$\text{ad } e_{s\pm r}$$

for the positive roots (pseudo as well as proper) and finally for the negative root and the "Cartan element"

$$\text{ad } e_{-r} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -N_{s,-r} & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\text{ad } h = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{49}$$

It is now straightforward to compute the normalization polynomial p, and we get

$$p(\bar{\zeta}, \zeta') = 1 - 2\zeta'_r \bar{\zeta}_r + \zeta'^2_r \bar{\zeta}^2_r \tag{50}$$

with the coherent states being [$v = (0, 1, 0, 0, 0, 0) = |2\rangle$, just as for A_1, upon which this algebra is built after all]

$$|\zeta\rangle \begin{pmatrix} -\zeta_r^2 \\ 1 \\ 0 \\ -\frac{1}{3}\zeta_s\zeta_r^2 N_{s,r} \\ \zeta_s N_{s,-r} \\ \zeta_r \end{pmatrix} \tag{51}$$

Notice that p is independent of ζ_s.

Let us summarize our experiences with nonsemisimple Lie algebras so far. First we have noticed that central elements will not contribute to the ad e or ad f terms, but at most through the commutators, i.e., only if they can be written as $c = [g_1, g_2]$, $g_1, g_2 \in \mathfrak{g}$ (hence if and only if the central element c lies in the derived subalgebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$). Second we notice that the normalization polynomial need not depend on all the variables ζ_α. The example of the "fan algebra" $\mathfrak{f}_3$ showed this quite clearly. The normalization polynomial will, however, always depend on *all* the proper roots, since these span a semisimple subalgebra. In general, variables ζ_α corresponding to $\mathfrak{g}_\alpha \subset \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ will contribute, unless of course $\mathfrak{g}_\alpha \subseteq Z(\mathfrak{g})$, where $Z(\mathfrak{g})$ denotes the center of the Lie algebra. For a semisimple algebra, $\mathfrak{g}' = \mathfrak{g}$ and $Z(\mathfrak{g}) = 0$, so all variables will appear.

2.3. Loop and Kac–Moody Algebras

Since this construction is based directly on the roots and the corresponding structure constants and Cartan matrices it is quite natural to attempt an extension to Kac–Moody algebras. Recall (Kac, 1990) that these can be defined in terms of generalized Cartan matrices as follows. An $n \times n$ matrix A is called a generalized Cartan matrix if it satisfies

$$A_{ii} = 2, \qquad A_{ij} \in -\mathbb{N}_0, \qquad A_{ij} = 0 \Rightarrow A_{ji} = 0, \qquad i, j = 1,2, \ldots, n$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}$ is the set of nonnegative integers. For the n primitive roots α_i (i.e., the ones spanning the entire root space) the algebraic relations are then

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} h_i \\ [h_i, e_j] &= A_{ij} e_j \\ [h_i, f_j] &= -A_{ij} f_j \\ [h_i, h_j] &= 0 \end{aligned}$$

with $e_i = e_{\alpha_i}, f_i = f_{\alpha_i}$, and h_i elements of the Cartan subalgebra $\mathfrak{h}$, $h_i = \langle \alpha_i, h \rangle$, $h \in \mathfrak{h}$.

Furthermore, for the particularly simple case of affine Kac–Moody algebras, the set of imaginary roots becomes very simple, namely $\Delta_{\text{im}} = \mathbb{Z}\delta = \{0, \pm n\delta | n = 1, 2, \ldots\}$. Such infinite-dimensional Lie algebras can be represented as central extensions of loop algebras. Thus it seems advantageous to begin by considering loop algebras.

Given a finite-dimensional Lie algebra, semisimple or not, $\mathfrak{g}$, we form its loop algebra $\mathfrak{g}_{\text{loop}} := C^\infty(S^1) \otimes \mathfrak{g}$ by defining the generators $e^n_\alpha = e_\alpha z^n, f^n_\alpha = f_\alpha z^n, h^n_i = h_i z^n$, where e_α, f_α, h_i are the generators of $\mathfrak{g}$ and where $z \in S^1$ (i.e., $z \in \mathbb{C}$ with $|z| = 1$). If g_1, g_2 are two arbitrary elements of $\mathfrak{g}$, then we define $[g^n_1, g^n_2] = z^{n+m} [g_1, g_2]$, where $g^n_i = g_i z^n$.

Now, in this case we can define $x(\zeta)$ as

$$x(\zeta) = \exp\left(\sum_{n=-\infty}^{\infty} \sum_{\alpha} \zeta_{\alpha,n} \text{ ad } e_{\alpha}^{n}\right) = \exp\left(\sum_{\alpha} \zeta_{\alpha}(z) \text{ ad } e_{\alpha}\right) \tag{52}$$

where we have defined

$$\zeta_{\alpha}(z) := \sum_{n=-\infty}^{\infty} \zeta_{\alpha,n} z^{n} \tag{53}$$

Hence ζ_{α} becomes an analytic function $S^1 \to \mathbb{C}$. If $|\zeta\rangle$ is a coherent state for $\mathfrak{g}$, then $|\zeta(z)\rangle$ is a coherent state for the corresponding loop algebra $\mathfrak{g}_{\text{loop}}$, and we define the inner product to be

$$\langle\zeta(z)|\zeta'(z')\rangle = \int_{S_1} \langle\zeta(z)|\zeta'(z')\rangle_0 \delta(z, z')\, dz\, dz' \tag{54}$$

where $\langle\cdot|\cdot\rangle_0$ denotes the inner product in $\mathfrak{g}$, i.e., ignoring the dependence on z, z'. Thus p, the normalization polynomial, becomes a functional of $\zeta_{\alpha}(z) \in C^{\infty}(S^1) \otimes \mathbb{C}$. Explicitly,

$$p[\bar{\zeta}, \zeta'] := \int_{S_1} p_0(\bar{\zeta}(z), \zeta(z))\, dz \tag{55}$$

where p_0 denotes the normalization polynomial of $\mathfrak{g}$.

An affine Kac–Moody algebra is, as already mentioned, a nontrivial central extension of a loop algebra. If $\mathfrak{g}$ denotes a finite-dimensional Lie algebra, then the corresponding Kac–Moody algebra is $\hat{\mathfrak{g}}_k := \mathfrak{g}_{\text{loop}} \oplus K\mathbb{F}$, where K is the central element and k is its eigenvalue. As we saw in Section 2, central extensions lead to very small modifications of the coherent-states. We then get

$$|\zeta, z\rangle = |\zeta(z)\rangle + c(\zeta)|K\rangle \tag{56}$$

$$p_k[\bar{\zeta}, \zeta'] = \int_{S_1} (p_0(\bar{\zeta}(z), \zeta'(z)) + c^*(\bar{\zeta})c(\zeta'))\, dz \tag{57}$$

for a general affine Kac–Moody algebra.

Furthermore, using the general relationship for central extensions (20), we have

$$c(\zeta) = c(\zeta, v_0) := \sum_{m,n\in\mathbb{Z}} \sum_{\alpha,\beta\in\Delta_+} \zeta_{\alpha,m} z^{m} c_{\alpha\beta}^{mn} v_{n}^{\beta} := k\left(z \frac{d}{dz}\zeta\bigg|v_0\right) \tag{58}$$

where $c^{mn}_{\alpha\beta}$ are the structure coefficients,

$$[e^m_\alpha, e^n_\beta] = N_{\alpha,\beta} e^{m+n}_{\alpha+\beta} + c^{mn}_{\alpha\beta} K \tag{59}$$

i.e.,

$$c^{mn}_{\alpha\beta} = km\delta_{m,-n}\, \kappa_{\alpha\beta} \tag{60}$$

where $\kappa_{\alpha\beta} = (\alpha|\beta)$ is the inner product in root space. We have also defined $v^\beta_n = v_\beta, \forall n$.

For nonaffine Kac–Moody algebras not much is known, but we can still attempt to use our constructive procedure. The set of imaginary roots becomes more complicated now. But we can write (Kac, 1990)

$$\Delta_{\text{im}} = \bigcup_{w \in W} w(\mathcal{K}) \tag{61}$$

where W is the Weyl group and $\mathcal{K}$ is some subset of the root lattice. So the basic quantity $x(\zeta)$ gets modified accordingly to

$$x(\zeta) = \exp\left(\sum_{\alpha\in\Delta^+_{re}}\left[\zeta_\alpha \text{ ad } e_\alpha + \sum_{I,\alpha_I\in\kappa}\sum_{w\in W} \varepsilon(w, I)\zeta_{w(\alpha_I),\alpha} \text{ ad } e_{w(\alpha_I)}\right]\right) \tag{62}$$

where $\varepsilon(w, I)$ is some number taking care of the possible multiplicity. In concrete cases one will then often be able to write ζ as a function $\zeta(z)$ with z in some set. But since we do not have any more concrete definition of either $\mathcal{K}$, W, or Δ_{im}, we will not be able to do more here.

As a final comment, $x(\zeta)$ for Kac–Moody algebras is closely related to (generalized) screening operator (Fuchs, 1992). One considers an algebra with generators e_α, f_α, h_i as usual, in some representation (always the adjoint representation in our case, just some formal representation in conformal field theory, CFT). Let $\langle\lambda|$ be a lowest weight vector in the appropriate module; then

$$\langle\lambda| e^{\Sigma_\beta x^\beta e_\beta} e^{te_\alpha} = \langle\lambda| e^{\Sigma_\beta (x^\beta + V^\beta_\alpha(x)t + O(t^2))e_\alpha}$$

$$\langle\lambda| e^{-te_\alpha} e^{\Sigma_\beta x^\beta e_\beta} = \langle\lambda| e^{tS_\alpha + O(t^2)} e^{\Sigma_\beta x^\beta e_\beta}$$

where S_α is the screening operator, $S_\alpha(x) = S^\beta_\alpha(x)\partial_\beta$, where $S^\beta_\alpha = -V^\beta_\alpha + f_{\gamma\alpha}{}^\beta x^\gamma$. The quantity V^β_α is the vertex operator. These operators play a crucial role in conformal field theory, in the construction of free field representation.

3. DIFFERENTIAL OPERATOR AND FREE FIELD REALIZATIONS

By construction, the algebra $\mathfrak{g}$ acts on the space $\mathcal{H}(\mathfrak{g})$ of coherent states $|\zeta\rangle$. Since this space $\mathcal{H}$ is a space of (vector-valued) functions, $\mathcal{H}(\mathfrak{g}) \subseteq$

$\mathbb{F}((\zeta)) \otimes \mathbb{F}^d$, $d = \dim \mathfrak{g}$, it is natural to look for realizations of $\mathfrak{g}$ in terms of differential operators. Define $\partial_\alpha = \partial/\partial\zeta^\alpha$, we then look for quantities E_α, F_α, H_i satisfying

$$E_\alpha(\zeta, \partial)|\zeta\rangle := \text{ad } e_\alpha|\zeta\rangle \tag{63}$$

$$F_\alpha(\zeta, \partial)|\zeta\rangle := \text{ad } f_\alpha|\zeta\rangle \tag{64}$$

$$H_i(\zeta, \partial)|\zeta\rangle := \text{ad } h_i|\zeta\rangle \tag{65}$$

We can find these quantities by using the BCH formula. Consider the corresponding Chevalley generators $x_\alpha(t) = \exp(t \text{ ad } e_\alpha)$, $x_{-\alpha}(t) = \exp(t \text{ ad } f_\alpha)$, $x_i(t) = \exp(t \text{ ad } h_i)$, and notice that

$$x_\alpha(t)x(\zeta) := e^{t \text{ ad } e_\alpha} e^{\Sigma_{\beta>0}\zeta^\beta \text{ad} e_\beta} = e^{\Sigma_{\beta>0}\zeta^\beta \text{ad} e_\beta + t\Sigma_\beta V_\alpha^\beta(\zeta)\text{ad} e_\beta + O(t^2)} \tag{66}$$

implies

$$E_\alpha(\zeta, \partial) = \sum_\beta V_\alpha^\beta(\zeta)\partial_\beta \tag{67}$$

Awata *et al.* (1991), Feigin and Frenkel (1990), and Bouwknegt *et al.* (1990a,b) give the "vertex operator" V_α^β in terms of the structure coefficients $f_{\alpha\beta}^\gamma$; we want to find an expression solely in terms of $N_{\alpha,\beta}$ and the Cartan matrix, which are the appropriate quantities to use for a Chevalley basis. From the definition it follows that

$$V_\alpha^\beta = \delta_\alpha^\beta + \frac{1}{2}\sum_\gamma N_{\gamma,\alpha}\delta_{\alpha+\gamma}^\beta\zeta_\gamma - \frac{1}{4}\sum_{\gamma,\delta} N_{\gamma,\alpha}N_{\alpha+\gamma,\delta}\delta_{\alpha+\gamma+\delta}^\beta\zeta_\gamma\zeta_\delta + \dots \tag{68}$$

We will write this as

$$V_\alpha^\beta = \delta_\alpha^\beta \sum_{n\geq 1} M_n C_{\alpha;\alpha_1\dots\alpha_n}^\beta \zeta_{\alpha_1} \dots \zeta_{\alpha_n}, \qquad \alpha, \beta \in \Delta_+ \tag{69}$$

in analogy with the notation of Bouwknegt *et al.* (1990a, b). Straightforward induction shows (the B_n are the Bernoulli numbers)

$$M_n = (-1)^n \frac{B_n}{n!} \tag{70}$$

$$\mathscr{C}_{\alpha;\alpha_1\dots\alpha_n}^\beta = \delta_{\alpha+\Sigma\alpha_i}^\beta N_{\alpha_n,\alpha}N_{\alpha_{n-1},\alpha+\alpha_n} \dots N_{\alpha_1,\alpha+\alpha_2+\dots+\alpha_n} \tag{71}$$

This follows from the following version of the BCH formula:

$$e^A e^{tB} = \exp\left(A + t\sum_{n=0}^{\infty} M_n(\text{ad}_A)^n B + O(t^2)\right)$$

which is easily proven.

Similarly, we get

$$F_\alpha = \sum_\beta V^\beta_{-\alpha}\partial_\beta + \sum_{i=1}^{l} P^i_{-\alpha}\lambda_i \tag{72}$$

$$H_i = \sum_\beta V^\beta_i\partial_\beta + \lambda_i \tag{73}$$

Such quantities have been introduced in the study of conformal field theories (CFTs; (Fuchs, 1992; Awata *et al.*, 1991; Feigin and Frenkel, 1990); the only new things here are the use of the adjoint representation, the new coherent states following from this, and finally the use of the structure coefficients of the Chevalley basis, $N_{\alpha,\beta}$, and the Cartan matrix, $A_{i\alpha}$.

Combining the results from Bouwknegt *et al.* (1990a,b), with our reformulation in terms of $N_{\alpha,\beta}$ we get

$$V^\beta_{-\alpha} = \sum_n N_n C^\beta_{-\alpha;\alpha_1\ldots\alpha_n}\zeta_{\alpha_1}\ldots\zeta_{\alpha_n} \tag{74}$$

$$V^\beta_i = -(\alpha^V_i|\beta)\zeta_\beta \tag{75}$$

$$P^i_{-\alpha} = \sum_{n\geq 1}\frac{1}{n!}C^i_{-\alpha;\alpha_1\ldots\alpha_n}\zeta_{\alpha_1}\ldots\zeta_{\alpha_n} \tag{76}$$

with $\alpha^V = 2\alpha/(\alpha|\alpha)$ the co-root of α and $(\cdot|\cdot)$ denoting the inner product in root space.

The explicit forms for the coefficients N_n and the $\mathscr{C}$'s are

$$N_n = \sum_{k=0}^{n-\mu}\frac{B_k}{k!(n-k)!} \tag{77}$$

$$\mathscr{C}^\beta_{-\alpha;\alpha_1\ldots\alpha_n} = \begin{cases} \delta^\beta_{\alpha_1+\ldots+\alpha_n-\alpha}N_{\alpha_n,-\alpha}N_{\alpha_{n-1},\alpha_n-\alpha}\ldots N_{\alpha_1,\alpha_2+\ldots+\alpha_n-\alpha} \\ \quad\text{if}\quad \exists i:\quad \sum_{j=n-i}^{n}\alpha_j - \alpha = 0 \\ \delta^\beta_{\alpha_1+\ldots+\alpha_{n-i-1}}N_{\alpha_n,-\alpha}\ldots N_{\alpha_{n-i-1},\alpha_n+\ldots+\alpha_{n-i-2}-\alpha} \\ \quad\times\sum_{j=1}^{l}\alpha^j A_{j,\alpha_{n-i-1}}N_{\alpha_{n-i-2},\alpha_{n-i-1}}\ldots N_{\alpha_1,\alpha_2+\ldots+\alpha_{n-i-1}} \\ \quad\text{if}\quad \exists i:\quad \sum_{j=n-i}^{n}\alpha_j - \alpha = 0 \end{cases} \tag{78}$$

$$\mathscr{C}^i_{-\alpha;\alpha_1\ldots\alpha_n} = -\frac{1}{n!}\alpha^i\delta_{\alpha_1+\alpha_2+\ldots+\alpha_n,\alpha}N_{\alpha_n,-\alpha}\ldots N_{\alpha_2,\alpha_3+\ldots+\alpha_n-\alpha} \tag{79}$$

$$\mathscr{C}^\beta_{i;\alpha_1\ldots\alpha_n} = -\delta^\beta_{\alpha_1+\ldots+\alpha_n}A_{i\alpha_n}N_{\alpha_{n-1},\alpha_n}\ldots N_{\alpha_1,\alpha_2+\ldots+\alpha_n}$$

with $\mu = \mu(-\alpha, \beta_1, \ldots, \beta_n)$ being the smallest integer such that $-\alpha + \beta_1 + \ldots + \beta_n \in \Delta_+$.

We can use our coherent states to reexpress these results. Introduce first of all the *deformed addition* in ζ-space, $(\zeta, \zeta') \mapsto \zeta \oplus \zeta'$, where $\zeta \oplus \zeta'$ is defined by

$$x(\zeta)x(\zeta') := x(\zeta \oplus \zeta') \tag{80}$$

The difference between $\zeta + \zeta'$ and $\zeta \oplus \zeta'$ only shows up in the nonprimitive roots, where the BCH theorem gives correction terms. For the examples of finite-dimensional semisimple Lie algebras of rank at most two we get the following explicit results:

$\mathfrak{g} \simeq A_1$:

$$\zeta \oplus \zeta' = \zeta + \zeta'$$

$\mathfrak{g} \simeq A_2$:

$$(\zeta \oplus \zeta') = \begin{pmatrix} \zeta_r + \zeta'_r \\ \zeta_s + \zeta'_s \\ \zeta_{r+s} + \zeta'_{r+s} + \frac{1}{2}N_{r,s}(\zeta_r\zeta'_s - \zeta_s\zeta'_r) \end{pmatrix}$$

$\mathfrak{g} \simeq B_2$:

$$(\zeta \oplus \zeta') = \begin{pmatrix} \zeta_r + \zeta'_r \\ \zeta_s + \zeta'_s \\ \zeta_{r+s} + \zeta'_{r+s} + \frac{1}{2}N_{r,s}(\zeta_r\zeta'_s - \zeta_s\zeta'_r) \\ \zeta_{2r+s} + \zeta'_{2r+s} + \frac{1}{2}N_{r,r+s}(\zeta_r\zeta'_{r+s} - \zeta_{r+s}\zeta'_r) \\ + \frac{1}{12}N_{r,s}N_{r,r+s}(\zeta_r^2\zeta'_s - \zeta_s\zeta_r'^2) \end{pmatrix}$$

$\mathfrak{g} \simeq G_2$:

$$(\zeta \oplus \zeta') = \begin{pmatrix} \zeta_r + \zeta'_r \\ \zeta_s + \zeta'_s \\ \zeta_{r+s} + \zeta'_{r+s} + \frac{1}{2}N_{r,s}(\zeta_r\zeta'_s - \zeta_s\zeta'_r) \\ \zeta_{2r+s} + \zeta'_{2r+s} + \frac{1}{2}N_{r,r+s}(\zeta_r\zeta'_{r+s} - \zeta_{r+s}\zeta'_r) \\ + \frac{1}{12}N_{r,s}N_{r,r+s}(\zeta_r^2\zeta'_s - \zeta_s\zeta_r'^2) \\ \zeta_{r+2s} + \zeta'_{r+2s} + \frac{1}{2}N_{s,r+s}(\zeta_s\zeta'_{r+s} - \zeta_{r+s}\zeta'_s) \\ + \frac{1}{12}N_{r,s}N_{s,r+s}(\zeta_s^2\zeta'_r - \zeta_r\zeta_s'^2) \end{pmatrix}$$

This example shows how the noncommutativity of the algebra induces a deformation of the addition in ζ-space.

Let us go back to the definition of $V_\alpha^\beta(x)$. We have

$$e^{\Sigma_{\alpha\in\Delta_+}\zeta_\alpha \mathrm{ad} e_\alpha}\, e^{t\,\mathrm{ad}\, e_\beta} = e^{t\Sigma_{\gamma\in\Delta_+}V^\gamma_\beta(x)\partial_\gamma + O(t^2)}\, e^{\Sigma_{\alpha\in\Delta_+}\zeta_\alpha \mathrm{ad} e_\alpha} \tag{81}$$

From this we see

$$\sum_{\gamma\in\Delta_+} V^\gamma_\beta(x)\partial_\gamma|\zeta'\rangle = \frac{\partial}{\partial t}\bigg|_{t=0} x(\zeta)e^{t\,\mathrm{ad}\, e_\alpha}|(-\zeta)\oplus\zeta'\rangle \tag{82}$$

and the matrix elements then become

$$\langle\zeta''|\sum_{\gamma\in\Delta_+} V^\gamma_\beta(x)\partial_\gamma|\zeta'\rangle = \frac{\partial}{\partial t}\bigg|_{t=0} \langle\zeta''|\zeta\oplus\tau(t)\oplus(-\zeta)\oplus\zeta'\rangle \tag{83}$$

where $\tau_\alpha(t) = t\delta_{\alpha\beta}$. One should note that the deformed sum is associative but not in general commutative. We can also use the Chevalley involution to rewrite this as

$$\langle\zeta''|\sum_{\gamma\in\Delta_+} V^\gamma_\beta(x)\partial_\gamma|\zeta'\rangle = \langle\zeta''\oplus(-\bar{\zeta})|\frac{\partial}{\partial t}\bigg|_{t=0}|\tau(t)\oplus(-\zeta)\oplus\zeta'\rangle \tag{84}$$

which is somewhat more symmetrical.

We can also use our normalization polynomial p to write

$$\begin{aligned}\langle\zeta''|\sum_{\gamma\in\Delta_+} V^\gamma_\beta(x)\partial_\gamma|\zeta'\rangle &= \frac{\partial}{\partial t}\bigg|_{t=0} p(\bar{\zeta}'', \zeta\oplus\tau(t)\oplus(-\zeta)\oplus\zeta')\\ &= \frac{\partial}{\partial t}\bigg|_{t=0} p(\bar{\zeta}''\oplus(-\zeta), \tau(t)\oplus(-\zeta)\oplus\zeta')\end{aligned} \tag{85}$$

This is our final result. It gives an explicit, intrinsic expression for the matrix elements of the vertex operator in the space of generalized coherent states.

The differential operator realization we found here agrees with the usual one, as one can see by considering, for instance, the case of $\mathfrak{g} = A_1$, where we get

$$E = \frac{\partial}{\partial\zeta}, \qquad F = \zeta^2\frac{\partial}{\partial\zeta} - \zeta\lambda, \qquad H = -2\zeta\frac{\partial}{\partial\zeta} + \lambda \tag{86}$$

We will not list the realizations of the remaining semisimple Lie algebras of rank ≤ 2, A_2, B_2, G_2. Instead we will just consider one more example, namely the Heisenberg algebra h_1. In this case we get

$$p = \frac{\partial}{\partial\zeta_p} + \zeta_q\frac{\partial}{\partial\zeta_1}, \qquad q = \frac{\partial}{\partial\zeta_q} - \zeta_p\frac{\partial}{\partial\zeta_1}, \qquad i\hbar 1 = \frac{\partial}{\partial\zeta_1} \tag{87}$$

This is a slightly unexpected realization, but one quickly sees that it satisfies the correct commutator relations. On the subspace of $\mathbb{C}[[\zeta_p, \zeta_q, \zeta_1]]$, where

$(\partial/\partial\zeta_1) f(\zeta) = kf(\zeta)$ with k some constant and f an arbitrary function, we get the more familiar realization $p = \partial + \underline{\zeta}k$, $q = \bar{\partial} - \zeta k$, $i\hbar 1 = k$, where we have written $\zeta = \zeta_p$, $\underline{\zeta} = \zeta_q$ to emphasize the analogy with complex analysis. This particular realization also clearly shows the Heisenberg algebra as a central extension of an Abelian Lie algebra.

Once one has the analogy with creation and annihilation operators (the root decomposition) and furthermore the realization in terms of differential operators acting on some "Fock space" (through the coherent states), it is obvious to look for realizations in terms of quantum fields, too.

In analogy with CFT we will then look for free field realizations of $\mathfrak{g}$, i.e., look for (bosonic) fields $\phi_i(\xi)$ and (bosonic) ghosts $\beta_\alpha(\xi)$, $\gamma_\alpha(\xi)$, where $\xi \in \Gamma$ is an element in some parameter space Γ. These fields are then substituted for ∂_α, ζ^α, λ_i in the following way:

$$\partial_\alpha \mapsto \beta_\alpha(\xi), \qquad \zeta^\alpha \mapsto \gamma^\alpha(\xi), \qquad \lambda_i \mapsto \sqrt{t}\partial\phi_i(\xi) \tag{88}$$

where t is some real number and $\partial\phi$ denotes the derivative of ϕ with respect to ξ. Given some ordering $: \cdot :$, we then look for realizations

$$E_\alpha = :V_\alpha^\beta(\gamma(\xi))\beta_\beta(\xi): \tag{89}$$

$$F_\alpha = :V_{-\alpha}^\beta(\gamma(\xi))\beta_\beta(\xi): + P^j_{-\alpha}(\gamma(\xi))\sqrt{t}\partial\phi_j(\xi) + \mathcal{F}_\alpha(\gamma(\xi), \partial\gamma(\xi)) \tag{90}$$

$$H_i = :V_i^\beta(\gamma(\xi))\beta_\beta \text{ (gj)}: + \sqrt{t}\partial\phi_i(\xi) \tag{91}$$

the function $\mathcal{F}_\alpha$ is a possible anomalous term. For affine Kac–Moody algebras this construction is well known (Wakimoto realization), and in this case $\Gamma = \mathbb{C}$. The anomalous term $\mathcal{F}_\alpha$ is known to be (Bouwknegt *et al.*, 1990 a,b), for a primitive root α_i (the general result can be found in the reference)

$$\mathcal{F}_{\alpha_i} = \left(\frac{k+t}{(\alpha_i|\alpha_i)} - 1\right)\partial\gamma^{\alpha_i}(z) \tag{92}$$

The only difference in our case is the explicit appearance of the adjoint representation instead of some formal exponential, e^{xe_α}.

For finite-dimensional Lie algebras, we will expect $\dim \Gamma \leq 1$, i.e., we have a zero- or one-dimensional field theory. For nonaffine Kac–Moody algebras we would expect $\dim \Gamma \geq 2$, but we will not be able to prove this. Due to a lack of knowledge about nonaffine Kac–Moody algebras we will restrict ourselves to finite-dimensional Lie algebras, semisimple or not.

Consider then a finite-dimensional Lie algebra $\mathfrak{g}$. We want to write down a free field realization à la Wakimoto for this. The parameter space Γ will be taken to be the discrete set $\mathbb{Z}$, i.e., $\dim \Gamma = 0$. The analogy with the OPEs of the affine Kac–Moody algebra case is then

$$E_\alpha(n)E_\beta(m) = \delta_{nm}N_{\alpha,\beta}E_{\alpha+\beta} \tag{93}$$

$$E_\alpha(n)F_\alpha(m) = \delta_{nm}\alpha^i H_i(n) \tag{94}$$

$$H_i(n)H_j(m) = 0 \tag{95}$$

$$H_i(n)E_\alpha(m) = \delta_{nm}A_{i\alpha}E_\alpha(n) \tag{96}$$

$$H_i(n)F_\alpha(m) = -\delta_{nm}A_{i\alpha}F_\alpha(n) \tag{97}$$

and so on. A note about the notation: the δ_{nm} need not be the actual Kronecker delta, it is merely a "reproducing kernel" in the sense that it acts like a Kronecker delta

$$\sum_n f(n)\delta_{nm} = f(m) \qquad \forall f \tag{98}$$

just like the $(z - w)^{-1}$ in the affine Kac–Moody algebra acts like a Dirac delta function

$$\oint \frac{f(z)}{z - w}\frac{dz}{2\pi i} = f(w)$$

In analogy with the affine Kac–Moody algebra case we have not written the "nonsingular terms," i.e., the terms which are not proportional to $(\delta_{nm})^k$ for some $k > 0$.

Next, we want to introduce free "fields" (since dim $\Gamma = 0$ we are actually working with quantum mechanics rather than quantum field theory)

$$\partial_\beta \mapsto \beta_\beta(m), \qquad \zeta_\alpha \mapsto \gamma_\alpha(m), \qquad \lambda_i \mapsto \sqrt{t}\delta\phi_i(n) \tag{99}$$

where

$$\beta_\alpha(n)\gamma_\beta(m) = \delta_{nm}\delta_{\alpha,\beta} \tag{100}$$

$$\phi_i(n)\phi_j(m) = \kappa(\alpha_i^\vee|\alpha_j^\vee)\Delta_{nm} \tag{101}$$

$$\delta\phi_i(n) := \phi_i(n+1) - \phi_i(n) \tag{102}$$

$$\delta\Delta_{nm} = \phi_{nm} \tag{103}$$

Here κ is some constant.

The question is then whether anomalous contributions come into play like they do in the infinite-dimensional case. In fact they have to, for the very same reasons as in the infinite-dimensional case, namely because of the λ_i part of F_α, which becomes proportional to the bosonic field ϕ_i in the Wakimoto realization. An extra term is then needed to compensate for the $\phi_i\phi_j$ contribution to the OPEs, i.e., it must contain a $\delta\gamma$ contribution. Straightforward computation yields the same result as for the affine Kac–Moody algebra case, since this only uses the root decomposition.

4. CONCLUSION

We have generalized the notion of coherent states from the harmonic oscillator using an analogy with the GNS construction for C^*-algebras. The resulting procedure is constructive and allowed us to handle not only semisimple Lie algebras, but also nonsemisimple ones, even those corresponding to noncompact Lie groups such as $su(1, 1)$, $so(2, 1)$, $so(3, 1)$, etc. Furthermore, affine as well as nonaffine Kac–Moody algebras could be treated with this procedure, too.

The only ingredient in the procedure is the Lie algebra structure, put more precisely, a root decomposition, the structure constants $N_{r,s}$, and the Cartan matrix. The representation used was the natural one, i.e., the adjoint representation acting on the underlying vector space of the algebra.

In this way, a coherent state becomes a vector-valued function, and the set of these states are $\mathbb{C}((\zeta)) \otimes \mathbb{F}^d$, with $d = \dim \mathfrak{g}$, for finite-dimensional Lie algebras, whereas for affine Kac–Moody and loop algebras formed from some finite-dimensional Lie algebra $\mathfrak{g}$, the set of coherent states spans $C^\infty(S^1) \otimes \mathbb{C}((\zeta)) \otimes \mathbb{F}^d$, i.e., the corresponding loop space.

The advantage of the proposed construction is the nilpotency of the adjoint representation, for semisimple algebras, making the space of coherent states finite dimensional, namely simply $\mathbb{C}(\zeta) \otimes \mathbb{F}^d$.

We finally defined differential operator and free field realizations of the algebras in analogy with what is done for affine Kac–Moody algebras in conformal field theory.

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